How Small is Small? Student Reasoning with Approximations in Introductory Calculus and Physics

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How Small is Small?

Exploratory study on the ways

students orient to and reason with

approximation tasks

across calculus and physics contexts

Approximation!

We consider scenarios that involve neglecting quantities in order to produce (numerical) approximations

Examples include:

- Considering the influence of error resulting from various numerical methods
- Neglecting computationally complicated terms in physical and/or mathematical scenarios

Bi-coastal student sample

- 22 total students
- Enrolled in or had completed coursework for 2 semesters of calculus and 2 semesters of physics*
- Major:
  - (various types of) engineering (15)
  - physics (4),
  - chemistry (2),
  - earth sciences (1)*

Tasks

2 of the 5 tasks used are relevant to what we discuss today:

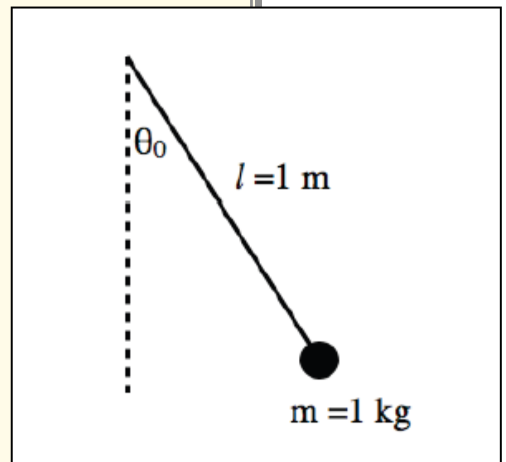
Task 1

The Taylor series about $x=0$ for $arctan(x)$ is given by

$$\arctan(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots$$

How big a value can x be, before stopping after the second term is a bad approximation?

You have a pendulum made of a metal ball on a string. The string is 1 meter long and the metal ball has a mass of 1 kg. You might know that the approximation for the period of a Pendulum for small oscillations is



$$T = 2\pi\sqrt{\frac{l}{g}}$$

where T is the period of the pendulum, l is the length of the pendulum, and g is acceleration due to gravity (9.81 m/s^2). This equation only holds for small angle oscillations of the pendulum. For larger angles, the period of a pendulum can be found with the following equation:

$$T = 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{1}{16}\theta_0^2 + \frac{11}{3072}\theta_0^4 + \ldots\right)$$

where θ_0 is the angle of displacement of the pendulum from vertical in <u>radians</u>. **You want to calculate the period of oscillation for this pendulum. How big can the angle of displacement of the pendulum be before the equation for small oscillations isn't a good approximation of the period?**

Task 2

A third task – for comparison

Imagine that you're a NASA scientist that is trying to make a calculation about how long it will take for a space ship fired from Cape Canaveral to reach the moon. You know what the force of the rocket engine will be (F_{rocket}). If you were to take a first step toward solving this problem, what factors would be important to consider?

Task 3

The additional tasks (not discussed today) were in the context of

-Newton's Method

-A velocity-time graph "driving problem"

Pay Attention! ...

to Contextual Differences

- –Attending to contextual differences
- –Examining contextual differences through student work on ARCTAN and PENDULUM
- –Themes of analysis
- –Examples of themes

Abstraction vs Contextual differences

- A goal of cross-disciplinary education:
  Students "transfer" their knowledge from one disciplinary context to another.

- Typically "...it is deemed important for learners to engage with multiple situations and to compare problem solutions in order to construct an abstract representation spanning them." (described in Lobato, 2006, p. 439)

- We argue: Instead of focusing on abstract knowledge, focus on contextual differences and the role they play in students' reasoning.

Abstraction vs Contextual differences

- Example: Watkins et al (2012) – A case study of students' views on the role of physics equations in biology:
  - A biology course focused on using ideas from physics to make sense of biologically relevant topics.
  - Ashlyn, on equations: "I don't like to think of biology in terms of numbers and variables. I feel like that's what physics and calculus is for…"

- The disciplinary context of the course is important for judging whether using equations seems appropriate/ desirable.

Watkins, Coffey, Redish, Cooke (2012). Disciplinary authenticity: Enriching the reforms of introductory physics courses for life-science students.

Abstraction vs Contextual differences

- Michael Wittmann –

  Sensitivity to how the question is asked

- John Thompson and Joe Wagner –

Some students (~20%) stated **integrals were different**,
but **works were equal**:
math: area under curve
physics: state-function / path-independence;
"assuming zero dissipative processes";
"assuming conservative force."

We argue …

- Use of Taylor series (for approximation) is an example of something which is not an abstract, domain-general skill that students apply across multiple contexts.

- Rather, contextual differences (related to different courses and disciplines) play a role in students' reasoning and orientation to these approximation tasks …

Research Focus

(1) How students reason differently about these approximation problems,

and

(2) The role of context (related to courses or disciplines) in explaining these differences, **and how these play out in students' decisions about approximation.**

Our Analysis

Starting with approximation, we examine:

- behaviors and strategies while problem solving
- explicit talk about the perceived commitments to or constraints of a particular discipline (physics, mathematics, engineering)

Approximation Themes

Students in our data largely made **approximation decisions** of the following types:

Decisions about …

(1) allowable tolerance and/or error

(2) the work (calculation, guessing and checking, etc…) required of a problem

(3) how to use what is "already known" about this situation

(4) how to know when "I'm done" with this problem

(5) "what counts" in the context of the problem

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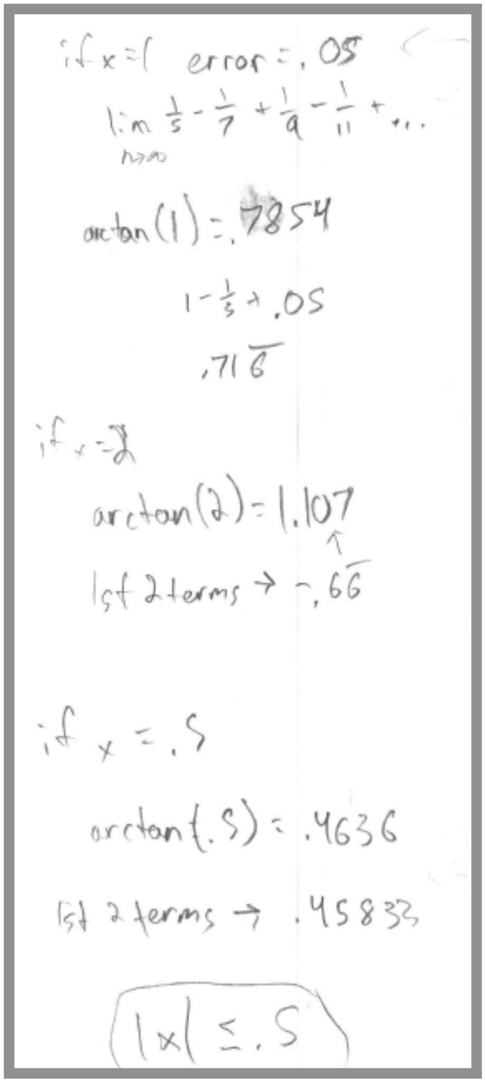
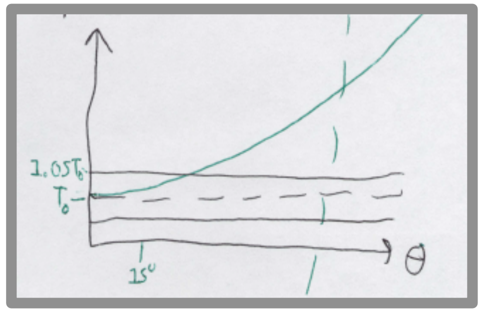
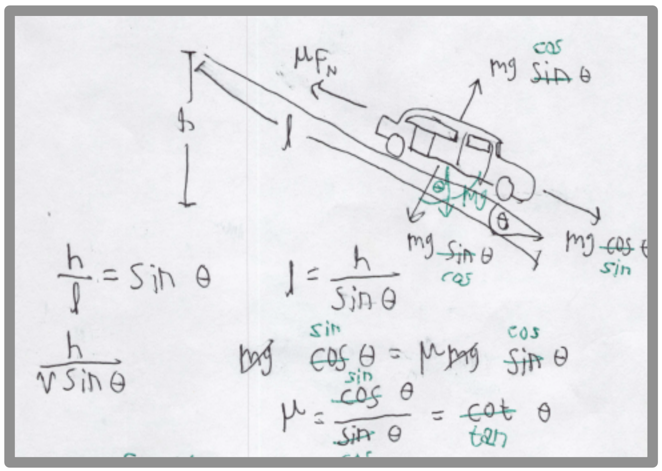
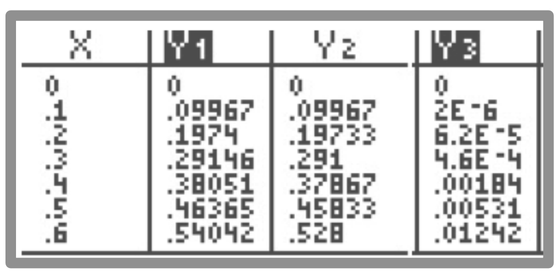
Students *did* productive things

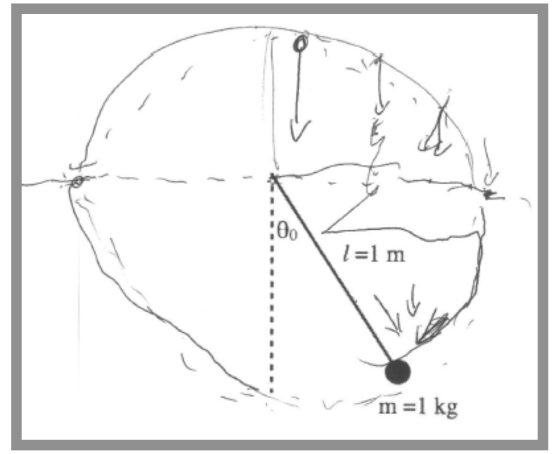


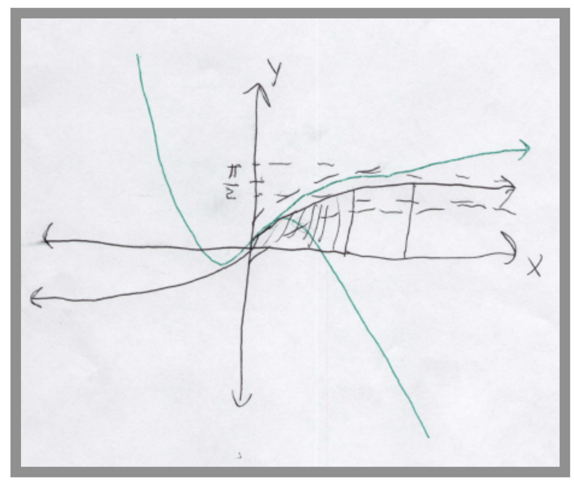






X	Y1	Y2	Y3
0	0	0	0
.1	.09967	.09967	2E-6
.2	.1974	.19733	6.2E-5
.3	.29146	.291	4.6E-4
.4	.38051	.37867	.00184
.5	.46365	.45833	.00531
.6	.54042	.528	.01242





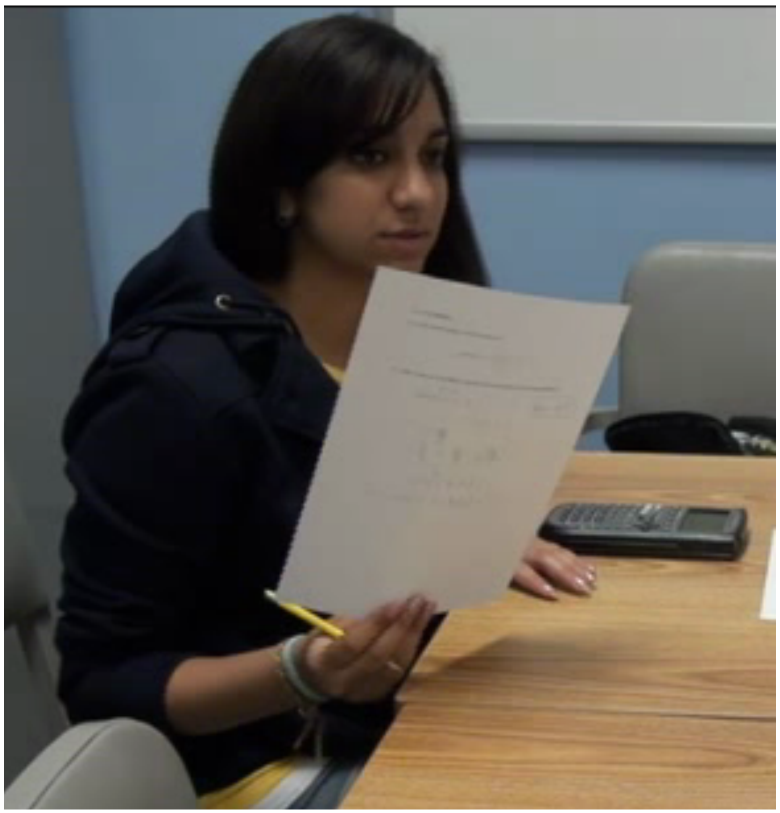
But today we'll talk about

what students *said*

instead of

what they *did* ...

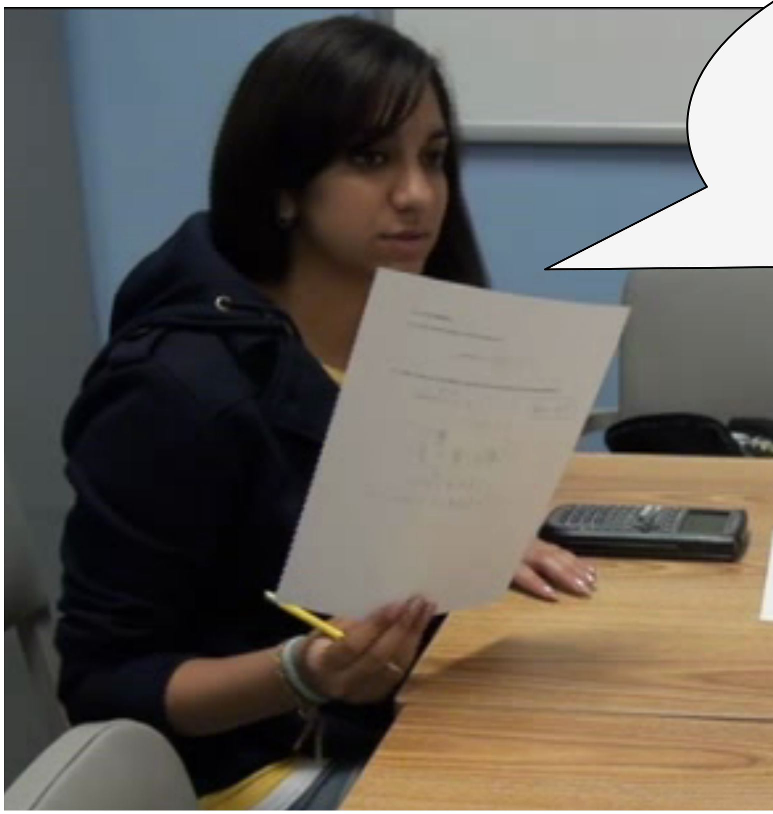
(1) Notion of "tolerances"



Joanne:
Comparing 'acceptable tolerance'
In ARCTAN and PENDULUM

(1) Notion of "tolerances"

[PENDULUM is] a real life problem … Because you're using this to model something that is actually happening, you might be using it in an engineering context. So [what] I would end up doing is for this one, I would have a smaller tolerance value



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1 of 2

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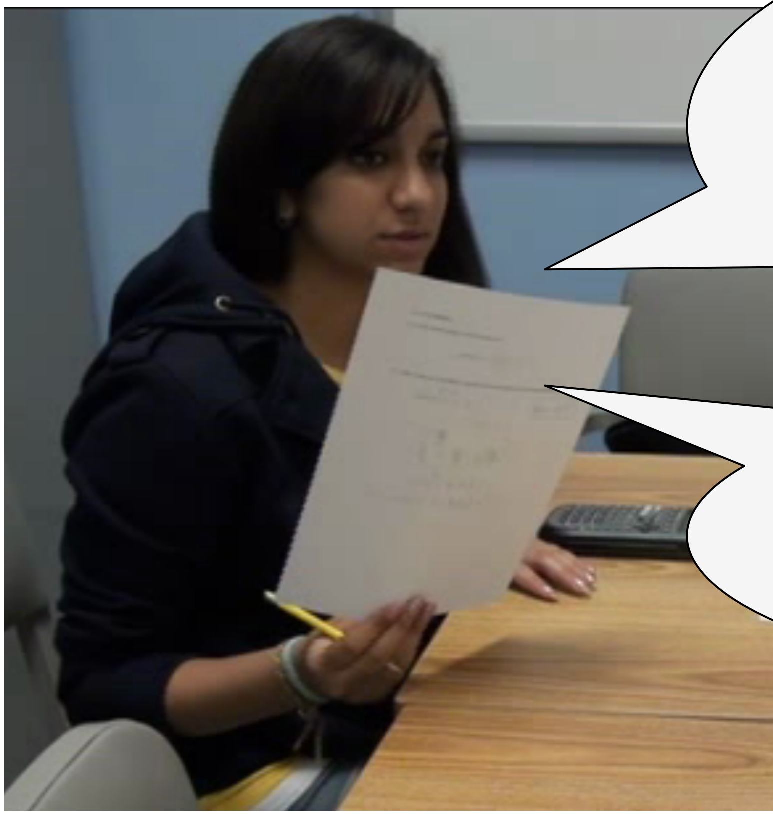
[ARCTAN tolerance] was ten to the negative 2 or negative 6 … so I would just make this ten to the negative 7 or 8. … I think it's more important to have a more accurate value.



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(1) Notion of "tolerances"

Brad:
While reasoning with ARCTAN,
which he identifies as a
"math problem"

(1) Notion of "tolerances"

… I'm an engineering major, so ... you're usually calculating to determine the tolerance or the stress levels or whatever of some material. Therefore I like to be super, super exact so I choose a really, really small tolerance. But, if this is just like a math problem, I mean you know, we are usually just given numbers and just work it out to get an answer.

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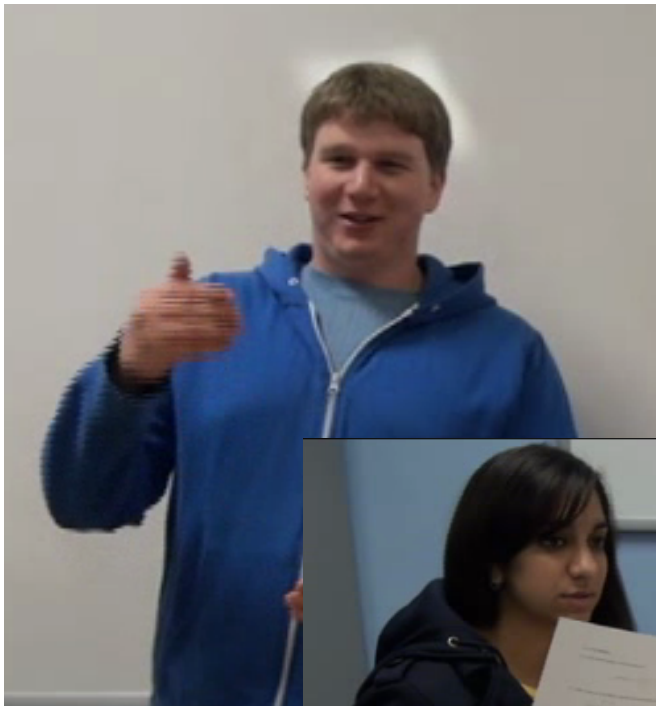
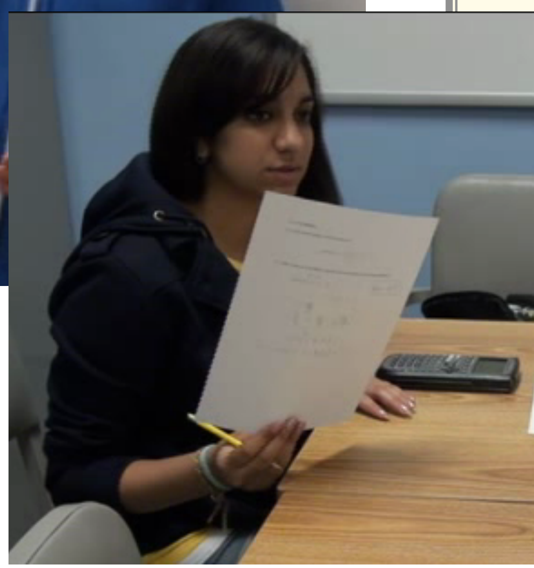
(1) Notion of "tolerances"





The interpretation of a problem as real-life/physics/engineering/math affects how small the tolerance needs to be.

Other students took a totally opposite stance ...


If it was left up to me what would be a good tolerance ... I mean to be honest if it were just a **normal math homework problem** and the tolerance were left up to be I would probably find a solution that has a very easy tolerance of some sort.

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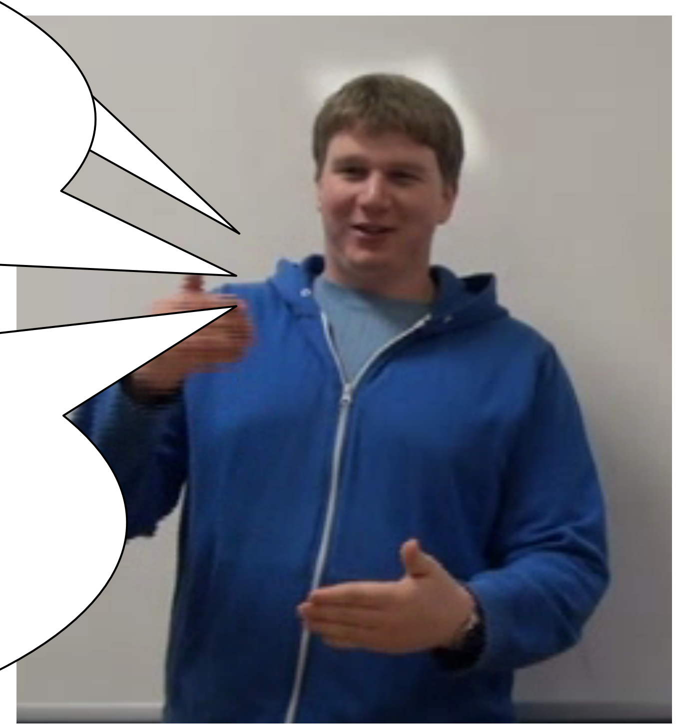
So like for instance like I dunno if you chose a tolerance of 1, which is abnormally large for these types of problems, you know, you would just solve to see you only need the first term to fit that.

But if it is more important to … have a certain type of accuracy and you are allowed a little leeway, so therefore it sort of throws more motivation to find something more exact with a better tolerance. And actually choose the smaller one.



(1) Notion of “tolerances”

The tolerance can be arbitrary on a homework exercise in a math course, but in situations where accuracy is important, then smaller tolerances are needed.

(2) Role of “calculation”

Will:
While solving ARCTAN, discussing
the strategy of
“plugging in values for *x*”

(2) Role of "calculation"

So in [PENDULUM], you sort of, you picked a number and you just plugged it in to see what you would get out.

Could you have done the same thing [on ARCTAN] and would that help?

Will:
While solving ARCTAN, discussing the strategy of "plugging in values for *x*"

(2) Role of "calculation"

You can…what I would have done is found where [x-x^3/3 = some decided on value]. And then found x's for that, but even then you can't really do that without a calculator.

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This, that's the thing, physics, you use a calculator. Math, you don't. So math you have to use the proper equations that you know to do the, find the answer. You cannot just guess.

Will:
While solving ARCTAN, discussing the strategy of "plugging in values for *x*"

(2) Role of “calculation”

Different kinds of solutions (“guess-and-check”, “compute-and-compare”, solving “proper equations”) are appropriate and allowed for physics vs math.

(5) "What Counts" for [*This Context*]

Nathan:
On ARCTAN, finds that x=1/2
gives an error of 3/1000s

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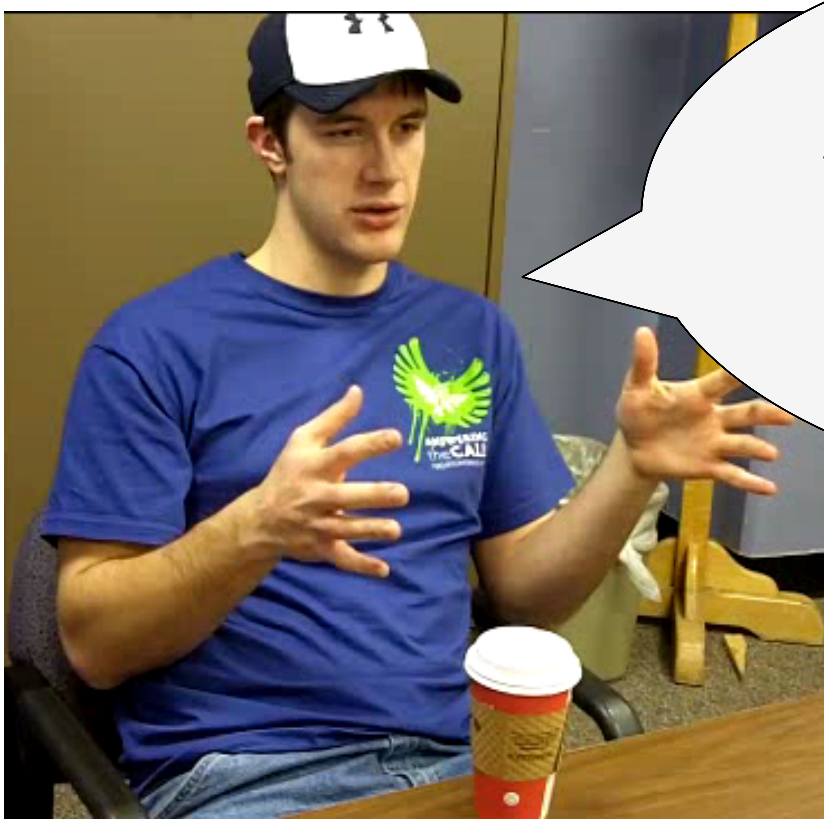

I'd say 3/1000s is a good approximation. Now another [thing] that I would consider is, if this is more than just strictly a math problem, because…depending on what the system was, that 3/1000s may make a pretty big difference.

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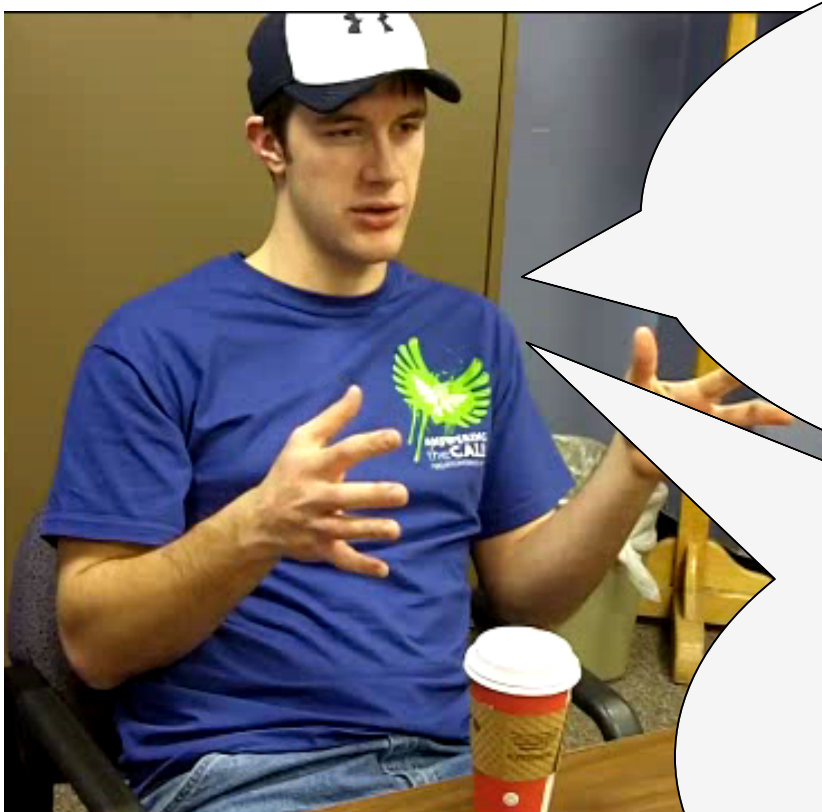

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…[For example] when you're talking about how many people got a certain disease or something like that, as you get into greater and greater numbers, it's still going to be the same percentage, I guess. But looking at raw, ok, who died, who didn't? [It's more people.]

(5) "What Counts" for [*This Context*]

Nathan:
Comparing how small is a good approximation on PENDULUM to his findings on ARCTAN?

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I think that increments of 5/100s of a second would be a pretty good approximation on the liberal end, because [of] the ability to get the data.

(5) “What Counts” for [*This Context*]

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I think that increments of 5/100s of a second would be a pretty good approximation on the liberal end, because [of] the ability to get the data.

On the other hand now, if you're trying to validate it against a stopwatch, for example. If you're trying to get a stopwatch, I mean, the human eye can see what, 10 frames per second? Something like that. So then, yeah it's a 10th of a second…

(5) "What Counts" for [*This Context*]

The meaning of the values produced when approximating depend on the purpose and context for which those numbers are interpreted.

Sooooo …

– Taylor series approximation is not an abstract, domain-general skill that students apply across multiple contexts.

– Rather, contextual differences, tied to the disciplines, play a role in students' reasoning **across all the various approximation decisions** that students made when solving these tasks.

A Next Step

As Michael Wittmann said yesterday:

"It's about what [students] think is the relevant activity and how they negotiate what to do next."

How students' **epistemologies** plausibly influence how they judge "what is a good approximation?," and make approximation decisions in these (and similar) problems

Where else can I see this awesome data?!

- RUME 2012 Proceedings:
  An evolving visual image of approximation with Taylor series
  [Champney & Kuo]

- PERC 2012 Proceedings:
  Considering factors beyond transfer in interdisciplinary research
  [Kuo, Champney, & Little]

- TRUSE 2012 Poster: tomorrow night!
  Disciplinary Dependence of Student Reasoning about Approximation [Champney & Kuo] – more on how disciplinary interpretations influence student reflections about "*what counts*"

And more to come!

Thank You!

The TRUSE 2010 organizers and NSF mini-grant

And for their valuable feedback …

- Physics Ed/Science Ed Group at Maryland, Andy Elby, Ayush Gupta, and Mike Hull
- Functions Group, Patterns Group, and Randi Engle at UC Berkeley

Questions?!

Bring it on!

Or think about it for awhile, and then email us:

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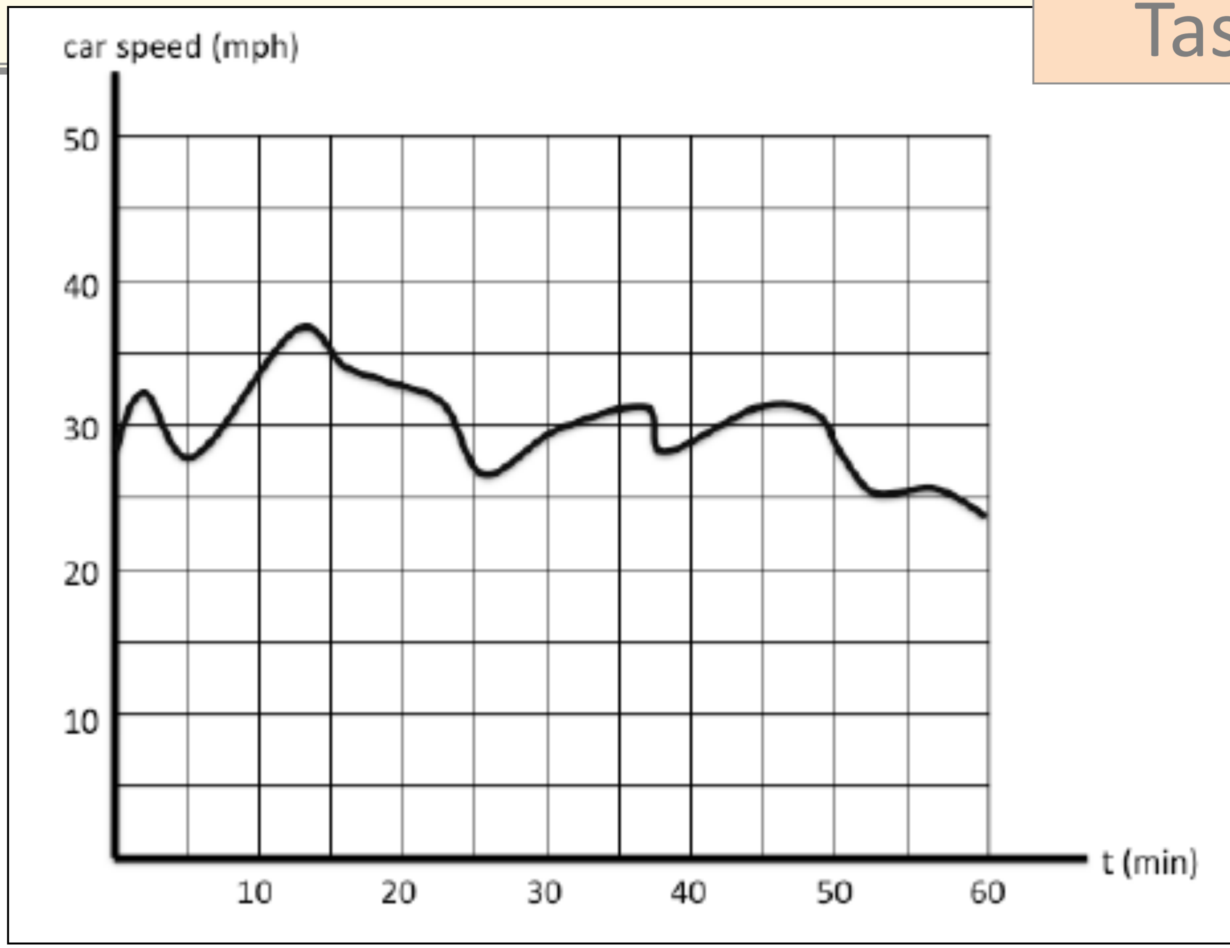
erickuo@umd.edu

little@berkeley.edu

You are driving on a lnog country road where the speed limit is 30 mph. The graph below shows the speed of the car over an hour. From this graph, how far does the car travel in that hour?

Task 4



Newton's Method is a way of approximating roots of a function that you can't nicely factor. For some function *f(x)*, one can approximate the roots by picking a starting point that is believed to be reasonably close to a root x_0, and running the following algorithm:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Task 5

Until the value for x_i converges. Then x_i is a root of *f(x)*.

A student is trying to find the roots of the following function:

$$f(x) = 7x^5 - x^4 + 3x^2 + 5x + 6$$

She correctly applied Newton's Method using $x_0=1$, stopping with x_{11}. Her results are below.
Given this data, would you say that the student did just enough, not enough, or too many iterations in order to find a root? Please explain your thinking.

$x_0 = 1$
$x_1 = 0.5238095238$
$x_2 = -0.4213109626$
$x_3 = -1.5316615789$
$x_4 = -1.2400986747$
$x_5 = -1.0299901928$
$x_6 = -0.9077016351$
$x_7 = -0.8674457933$
$x_8 = -0.8636626923$
$x_9 = -0.8636419374$
$x_{10} = -0.8636374634$
$x_{11} = -0.8636371324$